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On the non-existence of steady confined flows of a barotropic fluid in a gravitational field

By STEVE ARENDT
High Altitude Observatory & Advanced Study Program, National Center for Atmospheric Research, PO Box 3000, Boulder, CO 80307, USA

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We consider a semi-infinite (bounded above) plane-parallel layer of barotropic fluid in a constant gravitational field. We present a proof that flows of such a fluid cannot be time-independent in a reference frame wherein the flow's velocity field falls off asymptotically faster than the inverse of the radial distance, R. This includes all flows of finite kinetic energy as such flows must fall off faster than $R^{-1.5}$. The unsteadiness is due in part to the dynamical expansion of a compressible fluid in motion; this expansion leads to a density deficit so that in the presence of gravity the flow rises buoyantly and cannot be steady in time. The non-existence of certain classes of steady uniform-fluid flows is also discussed.

1. Introduction

Parker (1991) has recently pointed out that a barotropic fluid spinning in a vortical flow will exhibit a density deficit as compared to the ambient fluid outside the vortex. This deficit occurs because the centrifugal force on the spinning fluid reduces the pressure, and hence the density, of the fluid within the vortex. In the presence of a gravitational field, the resulting 'dynamical buoyancy' force will play a significant role in the dynamics of flows. For example, one would expect convection in a gravitationally stratified fluid (as in the sun) to be strongly influenced by this buoyancy force.

In a previous investigation, Arendt (1993*a*) studied the effect of the dynamical buoyancy on horizontal vortex tubes in a vertical gravitational field; the tubes were constructed by nesting a positive vorticity core inside a negative vorticity shell. It was shown that, in general, the positive vorticity core becomes displaced horizontally from its initial location, resulting in a vertical motion of the entire vortex tube through a dipole-like interaction between the core and shell. Surprisingly, while in all cases the dynamical buoyancy force is directed upward, the vertical motion of the tube can be either upward or downward, depending on the details of the vorticity distribution. If, however, the tube is constructed so as to have no initial outside velocity field, its motion is always upward.

This last fact is readily understood. If a flow is strictly confined, by which we mean that the flow velocity is non-zero only inside some closed region, then there are no forces to counter the upward dynamical buoyancy force and the volume containing the flow must rise. This leads one to suspect that there exist no steady strictly confined flows of a barotropic fluid in a gravitational field. Such a strictly confined flow would be buoyant, would rise, and would hence not be steady in time. In §2 we prove this assertion and in so doing remove two approximations used in the work of both Parker and Arendt. The first of these is that the flow size is small compared to the local density scale height, and the second is that the pressure and density variations caused by the flow are small compared to those caused by gravity.

To put this result into context, we note that steady strictly confined flows of uniform fluids do exist. In fact, in two dimensions, any circularly symmetric vortex having zero net circulation is confined and steady. Similarly, in three dimensions, confined and steady solutions are known to exist (Prendergast 1956; Moffat 1969; Low 1994). It is thus rather remarkable that such flows do not exist for a stratified barotropic fluid.

In §3, we relax the strict-confinement condition, requiring instead that the flow be asymptotically confined in the sense that the flow velocity vanishes at infinity. Taking a plane-parallel fluid layer with an upper boundary, we show by an extension of the proof in §2 that if the flow velocity falls off faster than R^{-1} and if $\boldsymbol{\omega}\cdot\hat{\boldsymbol{n}} = 0$ along the upper surface of the fluid layer, then the flow cannot be steady.

Before proceeding, we wish to clarify what is meant by a *steady* confined flow. We will call a flow steady and confined if it is not time-dependent in the frame of reference wherein the flow is confined (either strictly or asymptotically). Thus, for instance, Hill's vortex would not be confined and steady under our definition because in the reference frame in which its flow asymptotically vanishes, it is not steady; Hill's vortex is only steady in a frame in which its flow field asymptotically approaches a constant, that constant being the vortex's propagation velocity. Such uniformly propagating steady flows in a barotropic stratified fluid are not disallowed by the present proof. Indeed, such solutions have previously been constructed (Arendt 1993*c*).

2. Strictly confined flows

In this section, we consider strictly confined steady barotropic flows in a uniform gravitational field. By strictly confined we mean that the flow velocity is non-zero only inside a closed finite volume V embedded in an otherwise stationary body of fluid. On ∂V, the surface of V, we require that $\boldsymbol{u}\cdot\hat{\boldsymbol{n}} = 0$, where $\hat{\boldsymbol{n}}$ is the unit normal to the surface, and also that the fluid pressure be continuous. The fluid is first taken to be a barotrope of the form $P = C\rho^{\gamma}$ with $\gamma \geqslant 1$, but the proof is then shown to apply to any barotrope $P = h(\rho)$ with $\mathrm{d}^2h/\mathrm{d}\rho^2 \geqslant 0$.

We begin with the steady-state equations for an inviscid fluid in a uniform gravitational field

$$\nabla\cdot\rho\boldsymbol{u} = 0, \tag{2.1}$$

$$\rho(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = -\nabla P - \rho g\hat{\boldsymbol{z}}. \tag{2.2}$$

We will demonstrate that the only solution satisfying the aforementioned boundary conditions is the trivial one: $\boldsymbol{u} = 0$.

Consider first the region outside the flow volume. Denoting the pressure and density there by P_o and ρ_o respectively, (2.2) gives the hydrostatic law

$$-\nabla P_o - \rho_o g\hat{\boldsymbol{z}} = 0. \tag{2.3}$$

Using the barotropic relation $P = C\rho^{\gamma}$, we rewrite (2.3) to be

$$g = -\frac{\gamma P_o}{\rho_o^2}\frac{\mathrm{d}\rho_o}{\mathrm{d}z}. \tag{2.4}$$

Note that since $g > 0$, we have $\mathrm{d}\rho_o/\mathrm{d}z < 0$.

Now, inside the flow region introduce the notation

$$P = P_o + \Delta P, \tag{2.5}$$

$$\rho = \rho_o + \Delta\rho, \tag{2.6}$$

so that ΔP and $\Delta\rho$ are the pressure and density disturbances due to the flow; we do not assume that $\Delta\rho$ and ΔP are small. Substituting these into (2.2) and using (2.4) to replace g, we find

$$\rho(\boldsymbol{u}\cdot\boldsymbol{\nabla})\,\boldsymbol{u} = -\boldsymbol{\nabla}(\Delta P)+\Delta\rho\left(\frac{\gamma P_o}{\rho_o^2}\frac{\mathrm{d}\rho_o}{\mathrm{d}z}\right)\hat{z}, \tag{2.7}$$

where we have used (2.3) in writing (2.7). Multiplying the z-component of (2.7) by ρ_o^{-1} yields

$$\rho_o^{-1}\rho(\boldsymbol{u}\cdot\boldsymbol{\nabla})\,u_z = -\rho_o^{-1}\frac{\partial \Delta P}{\partial z}+\Delta\rho\left(\frac{\gamma P_o}{\rho_o^3}\frac{\mathrm{d}\rho_o}{\mathrm{d}z}\right), \tag{2.8}$$

which gives, upon rewriting,

$$\boldsymbol{\nabla}\cdot(\rho_o^{-1}\rho\boldsymbol{u}u_z)+\rho u_z^2\frac{1}{\rho_o^2}\frac{\mathrm{d}\rho_o}{\mathrm{d}z} = -\boldsymbol{\nabla}\cdot(\rho_o^{-1}\Delta P\hat{z})-\frac{1}{\rho_o^2}\frac{\mathrm{d}\rho_o}{\mathrm{d}z}\Delta P+\Delta\rho\left(\frac{\gamma P_o}{\rho_o^3}\frac{\mathrm{d}\rho_o}{\mathrm{d}z}\right). \tag{2.9}$$

Now, note that the barotropic relation $P = C\rho^{\gamma}$ can be rewritten as

$$\frac{\Delta P}{P_o} = \left(1+\frac{\Delta\rho}{\rho_o}\right)^{\gamma}-1. \tag{2.10}$$

Substituting this into (2.9), we find

$$\boldsymbol{\nabla}\cdot(\rho_o^{-1}\rho\boldsymbol{u}u_z)+\rho u_z^2\frac{1}{\rho_o^2}\frac{\mathrm{d}\rho_o}{\mathrm{d}z} = -\boldsymbol{\nabla}\cdot(\rho_o^{-1}\Delta P\hat{z})-\frac{P_o}{\rho_o^2}\frac{\mathrm{d}\rho_o}{\mathrm{d}z}\left(\left(1+\frac{\Delta\rho}{\rho_o}\right)^{\gamma}-1-\gamma\frac{\Delta\rho}{\rho_o}\right). \tag{2.11}$$

Integrating (2.11) over the flow volume V and using the boundary conditions to eliminate the surface integrals, we find

$$\int_V \frac{1}{\rho_o^2}\frac{\mathrm{d}\rho_o}{\mathrm{d}z}\left(\rho u_z^2+P_o\left[\left(1+\frac{\Delta\rho}{\rho_o}\right)^{\gamma}-1-\gamma\frac{\Delta\rho}{\rho_o}\right]\right)\mathrm{d}V = 0. \tag{2.12}$$

We now proceed to show that the term in square brackets, which we denote by F, is positive definite for $\gamma > 1$. First, the physically allowed range of $\Delta\rho/\rho_o$ is from -1 to ∞. Considering F to be a function of $\Delta\rho/\rho_o$, we find that $F = 0$ and $\mathrm{d}F/\mathrm{d}(\Delta\rho/\rho_o) = 0$ at $\Delta\rho/\rho_o = 0$. This is the only zero of the derivative of F within the allowed range of $\Delta\rho/\rho_o$, so F always has the same sign. To determine this sign, consider the point $\Delta\rho/\rho_o = -1$ at which $F = \gamma-1$. We conclude that $F \geqslant 0$ everywhere for $\gamma \geqslant 1$ and $F \leqslant 0$ everywhere for $\gamma < 1$.

Restricting our attention to the case $\gamma \geqslant 1$, we see that the integral in (2.12) is negative definite (recall $\mathrm{d}\rho_o/\mathrm{d}z < 0$). The only possible solution is then

$$u_z = 0, \tag{2.13}$$

$$\Delta\rho = \Delta P = 0. \tag{2.14}$$

Having $\Delta P = 0$, it is an easy matter to multiply the $\hat{x}$-component of (2.7) by x and integrate it over V to find

$$\int_V \rho u_x^2\,\mathrm{d}V = 0, \tag{2.15}$$

so that

$$u_x = 0. \tag{2.16}$$

Similar treatment of the $\hat{y}$-component yields

$$u_y = 0. \tag{2.17}$$

We conclude that there are no steady confined flows of a barotropic fluid (with $\gamma \geqslant 1$) in a uniform gravitational field.

It is straightforward to show that this proof holds for any barotropic relation $P = h(\rho)$ with $\mathrm{d}h/\mathrm{d}\rho > 0$ and $\mathrm{d}^2h/\mathrm{d}\rho^2 \geqslant 0$ everywhere. In this case the function in square brackets in (2.12) becomes

$$F = \frac{h(\rho_o+\Delta\rho)}{h(\rho_o)} - 1 - \frac{1}{h(\rho_o)}\frac{\mathrm{d}h(\rho_o)}{\mathrm{d}\rho_o}\Delta\rho. \tag{2.18}$$

To show that F is a positive definite, we begin by differentiating F with respect to $\Delta\rho$:

$$\frac{\mathrm{d}F}{\mathrm{d}\Delta\rho} = \frac{1}{h(\rho_o)}\left(\frac{\mathrm{d}h(\rho_o+\Delta\rho)}{\mathrm{d}(\rho_o+\Delta\rho)} - \frac{\mathrm{d}h(\rho_o)}{\mathrm{d}\rho_o}\right) = \frac{1}{h(\rho_o)}\left(\frac{\mathrm{d}h(\rho)}{\mathrm{d}\rho}\right)\Bigg|_{\rho=\rho_o}^{\rho=\rho_o+\Delta\rho} \tag{2.19}$$

Since $\mathrm{d}^2h(\rho)/\mathrm{d}\rho^2 > 0, \mathrm{d}F/\mathrm{d}\Delta\rho$ is only zero at $\Delta\rho = 0$. The function F is also zero at $\Delta\rho = 0$ so that F is of the same sign everywhere. To determine this sign, take the second derivative of F to obtain, at $\Delta\rho = 0$,

$$\left(\frac{\mathrm{d}^2F}{\mathrm{d}\Delta\rho^2}\right)_{\Delta\rho=0} = \frac{1}{h(\rho_o)}\frac{\mathrm{d}^2h(\rho_o)}{\mathrm{d}\rho_o^2}, \tag{2.20}$$

which is always greater than or equal to zero, by our assumption. Therefore, $F > 0$ everywhere and the proof that $\boldsymbol{u} = 0$ follows as before.

The curious restriction $\mathrm{d}^2h/\mathrm{d}\rho^2 \geqslant 0$ deserves some comment. First note that if the flow is sufficiently slow, then $\Delta\rho \ll \rho_o$ and $\Delta P \ll P_o$ so that the term in square brackets in (2.12) vanishes up to first order. The resulting integral is then positive definite, regardless of the properties of the barotropic relation. It is thus only for flows of higher velocities that this restriction obtains.

If we take the barotrope to be an isentrope, then the restriction becomes $(\mathrm{d}^2P/\mathrm{d}\rho^2)_S \geqslant 0$, where S is the entropy; this inequality is of some interest in the theory of shock waves. It is known that fluids normally obey the above inequality, the exception being in some region around a gas–liquid critical point (Zel'dovich & Raizer 1968).

3. Asymptotically confined flows

In this section, we relax the strict-confinement conditions and instead suppose that the flow is asymptotically confined, by which we mean that the flow velocity vanishes as one approaches infinity. We will show that a flow cannot be steady if the flow velocity vanishes at infinity faster than R^{-1}, R being the radial distance from the origin. The fluid is taken to be a plane-parallel layer with an upper surface and a constant gravitational field; the upper surface is required since the pressure of a barotropic fluid stratified by gravity in general becomes complex-valued at some height; an upper boundary below this height avoids this. An exception is an isothermal fluid which had a positive pressure everywhere. The surface can be either a free boundary upon which the pressure is constant or a rigid impenetrable boundary. We will enforce the condition $\boldsymbol{u}\cdot\hat{\boldsymbol{n}} = 0$, where $\hat{\boldsymbol{n}}$ is the unit normal to the surface, and also the condition $\boldsymbol{\omega}\cdot\hat{\boldsymbol{n}} = 0$, implying that no vortex lines end on the upper surface.

The proof proceeds in the same fashion as in the previous section, the difference being that the surface integrals obtained from integrating (2.11) require some consideration. To proceed, integrate (2.11) over the entire fluid region by employing a hemispherical volume (with the rounded part below and the upper surface corresponding with the upper boundary of the fluid layer) extending out to $R \to \infty$. We find

$$\int_V \frac{1}{\rho_o^2}\frac{d\rho_o}{dz}\left(\rho u_z^2 + P_o\left[\left(1+\frac{\Delta\rho}{\rho_o}\right)^\gamma - 1 - \gamma\frac{\Delta\rho}{\rho_o}\right]\right) dV = -\int_S \frac{1}{\rho_o}(\rho \boldsymbol{u} u_z + \Delta P \hat{\boldsymbol{z}})\cdot d\boldsymbol{S}. \quad (3.1)$$

We split the surface S into two pieces: S_1 and S_2, where S_1 is the upper boundary of the fluid layer and S_2 is a hemispherical surface extending out to $R \to \infty$.

Consider the integration over S_2, assuming that $\boldsymbol{u}$ falls off with distance. As $R \to \infty$, we have $\rho \to \rho_o$, so that $\rho \boldsymbol{u} u_z/\rho_o \approx \boldsymbol{u} u_z$. The surface integral of this quantity is zero if $\boldsymbol{u} \to 0$ faster than R^{-1} in three dimensions and $R^{-\frac{1}{2}}$ in two dimensions. Furthermore, if the velocity falls off this fast then it is straightforward to see from Euler's equation that $\Delta P/\rho_o$ drops off faster than R^{-2} in three dimensions and R^{-1} in two dimensions so that its integral over S_2 is also zero. To avoid tedious repetition, we quote only the three-dimensional result in the rest of the paper. Equation (3.1) then becomes, after enforcing $\boldsymbol{u}\cdot\hat{\boldsymbol{n}} = 0$ on S_1,

$$\int_V \frac{1}{\rho_o^2}\frac{d\rho_o}{dz}\left(\rho u_z^2 + P_o\left[\left(1+\frac{\Delta\rho}{\rho_o}\right)^\gamma - 1 - \gamma\frac{\Delta\rho}{\rho_o}\right]\right) dV + \int_{S_1} \frac{\Delta P \hat{\boldsymbol{z}}}{\rho_o}\cdot d\boldsymbol{S} = 0. \quad (3.2)$$

In the previous section we have shown that the volume integral above is negative definite. We now proceed to show that the surface integral is also negative definite. This arises essentially from Bernoulli's theorem employed along the upper boundary. To begin, write Euler's equation (2.2) as follows:

$$\nabla\left(\frac{u^2}{2}\right) + \boldsymbol{\omega}\times\boldsymbol{u} = -\frac{\nabla P}{\rho} - g\hat{\boldsymbol{z}} = -\nabla(f(\rho) - f(\rho_o)), \quad (3.3)$$

where

$$f(\rho) = \int \frac{1}{\rho}\frac{dh}{d\rho}\, d\rho, \quad (3.4)$$

and $P = h(\rho)$. We have used (2.3) in writing (3.3). Now, if $\boldsymbol{\omega}\cdot\hat{\boldsymbol{n}} = 0$ and $\boldsymbol{u}\cdot\hat{\boldsymbol{n}} = 0$ on S_1, then $\boldsymbol{\omega}\times\boldsymbol{u}$ has no component tangential to S_1, and we can integrate (3.3) along the surface to obtain

$$\tfrac{1}{2}u^2 + (f(\rho) - f(\rho_o)) = \text{constant}. \quad (3.5)$$

The constant is found to be identically zero from the fact that $\boldsymbol{u} \to 0$ and $\Delta P \to 0$ as $R \to \infty$ along the upper boundary. We then have

$$\tfrac{1}{2}u^2 + f(\rho) - f(\rho_o) = 0 \quad (3.6)$$

on S_1. This gives

$$f(\rho_o + \Delta\rho) - f(\rho_o) \leqslant 0. \quad (3.7)$$

Now, $df/d\rho = (1/\rho)\, dP/d\rho > 0$, so that (3.7) implies that $\Delta\rho < 0$ and hence $\Delta P < 0$. Using the additional fact that $\hat{\boldsymbol{z}}\cdot d\boldsymbol{S} > 0$, we conclude that the surface integral in (3.2) is negative definite. Hence the entire expression in (3.2) is negative definite and we must have

$$u_z = 0, \quad (3.8)$$

$$\Delta P = \Delta\rho = 0. \quad (3.9)$$

Having these, the equations for u_x and u_y are

$$(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = 0, \quad (3.10)$$

$$\nabla\cdot\boldsymbol{u} = 0. \quad (3.11)$$

The only solutions to these equations are flows with straight streamlines extending to infinity along which the flow velocity is constant. Since we have already excluded flows not vanishing at infinity, we exclude these solutions. Hence,

$$u_x = u_y = 0, \tag{3.12}$$

and our proof is complete.

In the proof, we required that $\boldsymbol{\omega}\cdot\hat{\boldsymbol{n}} = 0$ on the surface of the fluid layer in order to demonstrate that the pressure was negative everywhere on the surface. It is unclear if the theorem holds for the case where $\boldsymbol{\omega}\cdot\hat{\boldsymbol{n}} \neq 0$ on the surface.

Throughout the proof, we have not specified the complete nature of the boundary conditions on the upper boundary, but rather only that $\boldsymbol{u}\cdot\hat{\boldsymbol{n}} = 0$ and $\boldsymbol{\omega}\cdot\hat{\boldsymbol{n}} = 0$ there. The final result holds for a free boundary upon which the pressure is constant or a rigid impenetrable boundary. Furthermore, one could add vertical impenetrable sidewalls instead of enforcing the asymptotic decay of the flow. The proof is unaffected if $\boldsymbol{\omega}\cdot\hat{\boldsymbol{n}} = 0$ on the sidewalls, a condition that is required for the evaluation of the constant in the Bernoulli equation (3.5).

In fact, only the imposition of a lower boundary surface on the fluid will affect the proof, since such a boundary surface S_3 will introduce an additional surface integral onto the left-hand side of (3.2):

$$\int_{S_3} \frac{\Delta P}{\rho_o}\hat{\boldsymbol{z}}\cdot \mathrm{d}\boldsymbol{S}. \tag{3.13}$$

Using the Bernoulli theorem (3.6) to show that $\Delta P < 0$ (if $\boldsymbol{\omega}\cdot\hat{\boldsymbol{n}} = 0$ on S_3), as well as the fact that $\hat{\boldsymbol{z}}\cdot \mathrm{d}\boldsymbol{S} < 0$, we find that the above integral is positive definite. Thus, if the magnitude of this integral can be made to match the magnitude of the sum of integrals in (3.2), then a steady flow may exist. However, if the flow is sufficiently weak at the lower boundary so that this integral cannot balance the others in (3.2), then a steady flow cannot be achieved. As a special case, for a flow slow enough so that $\Delta P \ll P_o$ everywhere, it can be shown that $f(\rho) - f(\rho_o) \approx \Delta P/\rho_o$ so that (3.2) becomes

$$\int_V \frac{u_z^2}{\rho_o}\left(-\frac{\mathrm{d}\rho_o}{\mathrm{d}z}\right)\mathrm{d}V + \int_{S_1} \tfrac{1}{2}u^2\hat{\boldsymbol{z}}\cdot\mathrm{d}\boldsymbol{S} = -\int_{S_3}\tfrac{1}{2}u^2\hat{\boldsymbol{z}}\cdot\mathrm{d}\boldsymbol{S}, \tag{3.14}$$

again under the assumption that $\boldsymbol{\omega}\cdot\hat{\boldsymbol{n}} = 0$ at the boundaries. We see that in order for the flow to be steady, the flow velocity at the bottom of the fluid layer must in general be larger than that on the top of the layer.

It has been mentioned that flows which translate steadily in time exist. To see how these evade the present theorem, consider writing (3.1) in the translating reference frame. In this frame, the total velocity field $\boldsymbol{u}$ may be written as $\boldsymbol{u} = \boldsymbol{v} - \boldsymbol{U}$ where $\boldsymbol{U}$, the velocity of translation, has no component in the $\hat{\boldsymbol{z}}$-direction. If the velocity $\boldsymbol{v}$ drops off faster than R^{-2} (as would be the case if the flow had, for example, strictly confined vorticity), then ΔP drops off faster than R^{-2} and the surface integral over S_2 still vanishes. After enforcing the boundary condition $\boldsymbol{u}\cdot\hat{\boldsymbol{n}} = 0$ on the upper boundary S_1 of the fluid layer, we are left again with equation (3.2). The Bernoulli theorem in (3.6) becomes

$$\tfrac{1}{2}u^2 + f(\rho) - f(\rho_o) = \tfrac{1}{2}U^2, \tag{3.15}$$

where U is the velocity of translation. Assuming for the present discussion that $\Delta P \ll P_o$, we expand $f(\rho) - f(\rho_o)$ in (3.15) to find

$$\tfrac{1}{2}u^2 + \Delta P/\rho_o = \tfrac{1}{2}U^2, \tag{3.16}$$

so that (3.2) gives
$$\int_V \frac{u_z^2}{\rho_o}\left(-\frac{\mathrm{d}\rho_o}{\mathrm{d}z}\right)\mathrm{d}V = \int_{S_1} \tfrac{1}{2}(U^2-u^2)\,\mathrm{d}S. \tag{3.17}$$

A steady flow may exist if the integral on the right-hand side is of the same magnitude as that on the left-hand side.

4. Discussion

In this paper we first presented a proof that strictly confined flows (i.e. flows non-zero only in a finite volume within an otherwise static fluid layer) of a barotropic fluid in a gravitational field cannot be time-independent. In so doing, we removed restrictions used in previous work, most notably the assumption of a flow region small compared to the local density scale height. The non-existence may be attributed to the dynamical expansion of fluid within a flow, although this physical mechanism has been obscured in the proof. The expansion leads to a density deficit so that in the presence of a gravitational field, the buoyant fluid rises and a steady state cannot be achieved.

The proof was then extended to include asymptotically confined flows whose velocity fields drop off faster than R^{-1}. In this case, it was necessary to assume $\boldsymbol{\omega}\cdot\hat{\boldsymbol{n}} = 0$ on the surface of the fluid layer so that the excess surface pressure could be shown to be negative; it is unclear if this restriction is essential to the non-existence result. We may ask whether the unsteadiness of asymptotically confined flows is due to the dynamical buoyancy force or to the presence of an upper surface to the fluid layer. One clue is found from the case of an isothermal fluid which requires no upper boundary; there the unsteady nature of the flow may be unambiguously attributed to the dynamical buoyancy force, leading us to believe that for a more general barotrope having an upper surface, the dynamical buoyancy force contributes to the flow's unsteadiness. On the other hand, we will in a moment discuss applications to unsteady uniform-fluid flows in the presence of a rigid wall. In that case there is no gravitational field and hence no dynamical buoyancy force so that the unsteadiness of those flows is completely attributable to the presence of a wall on which $\boldsymbol{u}\cdot\hat{\boldsymbol{n}} = 0$ and $\boldsymbol{\omega}\cdot\hat{\boldsymbol{n}} = 0$.

In fact, both mechanisms contribute in general. To see this, consider the total vertical force balance on an asymptotically confined flow with a rigid upper surface. To begin, there are the excess external pressure forces acting on the flow. The excess pressure from above is negative owing to Bernoulli's theorem along the surface of the fluid layer, and the excess pressure from below is zero because the flow is required to asymptotically come to rest. Thus, the flow has a total upward force on it and is already out of force balance, even before we consider the buoyancy force which only exacerbates the situation by adding an additional upward force. Hence, either mechanism alone will cause the flow to be time-dependent, but in general they act in concert.

It appears that there are two crucial features to this non-existence result, the first being the lack of a lower boundary on the fluid. As discussed above, simple force balance for a steady flow demands that the excess buoyancy of the flow be offset by an enhanced external pressure force pushing down. The upper boundary alone is incapable of supplying such a force, but if a lower boundary is present, then the dynamical buoyancy of the flow might be balanced by a difference in the excess pressure forces applied to the upper and lower boundaries. Again, each excess pressure must be negative by Bernoulli's theorem (assuming $\boldsymbol{\omega}\cdot\hat{\boldsymbol{n}} = 0$ on each boundary), but the difference between the two can be of either sign. Hence steady flows with lower boundaries may exist.

The second crucial feature is the use of a barotrope, which essentially forces the pressure deficit of a flow to appear as a density deficit, rather than a temperature deficit, and leads to the 'dynamical buoyancy' force. It is possible that confined steady flows of a non-barotropic fluid might exist; the removal of the barotropic restriction removes to an extent the coupling between pressure and density, so that a reduction in pressure could result in a reduction in temperature rather than density. If the total vertical force acting on a flow can be reduced to zero, then a steady state might be achieved. In any case, the results of the present paper suggest that the balance, if achieved, would be delicate. It should be noted that such a flow would be required to satisfy an energy conservation equation in conjunction with the momentum and mass conservation equations.

It is a remarkable fact that a flow with a velocity falling off faster than R^{-1} in a stratified barotropic fluid cannot be in a steady state. This rate of asymptotic decay is not a severe restriction. For perspective, consider the kinetic energy of a flow: $E_k = \frac{1}{2}\int \rho u^2 \, dV$. In order for this to be finite, the flow velocity must asymptotically approach zero as $R \rightarrow \infty$. For a uniform fluid, the flow velocity must go faster than $R^{-1.5}$, while for a stratified fluid it must go faster still since the density increases with depth. As this asymptotic behaviour obeys the conditions of the theorem of §3, we conclude that flows of finite kinetic energy in a barotropic stratified fluid without a lower boundary cannot be steady (in the frame of reference wherein they are confined); the only flows of finite kinetic energy which can be steady are those which translate transverse to gravity at a steady rate, as do stratified-fluid vortex tubes (Arendt 1993*c*). This result may be rephrased as follows, keeping in mind our choice of reference frame and the restriction of finite total kinetic energy: the kinetic energy distribution of a barotropic stratified-fluid flow cannot be time-independent.

Consider the application of the theorem to steady vorticity distributions. The flow surrounding a strictly confined vortex (here, the vorticity is meant to be strictly confined, not the flow velocity) in a uniform fluid in three dimensions drops off as R^{-3}, while in a stratified isothermal fluid or $\gamma = \frac{3}{2}$ polytrope, it falls off at least that fast and sometimes faster, depending on the orientation of the vortex with respect to gravity (Arendt 1993*b*). We may suppose the rapid fall-off of flow velocity outside an isolated vortex to be a general feature true for any barotrope. The theorem of §3 then has the immediate consequence that any strictly confined vorticity distribution in a stratified barotropic fluid cannot be steady state in the frame in which its flow drops off asymptotically.

Consider next the consequences of these results for steady stratified-fluid convection. Of course, a convecting fluid cannot be strictly barotropic, but if the departures from a barotrope are supposed small, we may apply the results of the present paper. Take, then, a convection pattern divisible into cells on whose surfaces $\boldsymbol{u}\cdot\hat{\boldsymbol{n}} = 0$ and $\boldsymbol{\omega}\cdot\hat{\boldsymbol{n}} = 0$, and assume that these cells have vertical sides. (It is straightforward to extend the following discussion to cells having non-vertical sides.) We may equivalently consider one of the cells confined to a column of unvarying cross-section by rigid impenetrable walls. The results of the present paper apply, and we conclude that the flow cannot be steady unless a lower boundary is present and (if $\Delta P \ll P_0$) the lower-boundary surface integral in (3.14) is large enough in magnitude to balance the sum of the other integrals in (3.14). This can occur only if the flow velocity at the bottom of the cell is larger than that at the top. In connection with this, it is interesting to note the numerical convection simulations of Hurlburt, Toomre & Massaguer (1984). Their two-dimensional box has stress-free upper and lower boundaries and periodic sidewalls; their examples of steady flow form cells on whose boundaries $\boldsymbol{u}\cdot\hat{\boldsymbol{n}} = 0$. These conditions fulfil the requirements leading to (3.14). In the solutions for steady convection (figures 4*a* and 5*a* of Hurlburt *et al.*), one finds that the flow velocities on the upper and lower boundaries are comparable.

As a specific example, consider the solar granulation. Numerical simulations (Cattaneo *et al.* 1992; Stein & Nordlund 1989) suggest that granules only appear near the surface of the convection zone of the sun; beneath the layer of granules is a general gentle updraught punctured by occasional thin fast downdraughts. If we consider the granules to be a layer of convective cells lying upon an almost stationary body of fluid, then the theorem of the present paper implies that such a layer of convection cells cannot be steady since it has no lower boundary. Indeed, observations show that granules are strongly time-dependent, varying on timescales comparable to their turnover time (Bray, Loughhead & Durrant 1984). This application of the theorem must be regarded with some caution owing to the non-barotropic nature of the convection and the presence of turbulence beneath and within the layer of granules.

Finally, we point out that the proof in this paper can also be applied to certain classes of steady confined flows of a *uniform*-density fluid. To begin, we note that if the fluid is assumed to be uniform, then all that remains in (3.2) is the surface integral of pressure over the upper boundary of the fluid. Since this integral is negative definite if $\boldsymbol{\omega}\cdot\hat{\boldsymbol{n}} = 0$ and $\hat{\boldsymbol{z}}\cdot\boldsymbol{n} > 0$ on the boundary, we immediately conclude that no steady asymptotically confined (with $\boldsymbol{u}$ falling off faster than R^{-1}) uniform-fluid flows exist in the presence of a rigid wall. In terms of vortex dynamics, no strictly confined vorticity distribution can be steady in the presence of a rigid wall in the reference frame wherein its flow field is asymptotically confined. Finally, it has previously been noted (Arendt 1993*a*) that the equations governing steady axisymmetric uniform-fluid flow having only toroidal vorticity ($\boldsymbol{\omega} = \omega\hat{\boldsymbol{\phi}}$) are equivalent to those governing a $\gamma = 2$ barotropic fluid in two dimensions. Thus, no steady and confined (strictly, or asymptotically with a rate faster than $R^{-\frac{1}{2}}$) axisymmetric flows having only toroidal vorticity exist in a uniform fluid. As an example, no uniform-fluid vortex ring (without helicity) can be constructed which is confined and steady (i.e. does not propagate). For perspective, a steady strictly confined axisymmetric flow with helicity is known (Prendergast 1956; Moffatt 1969; Low 1994).

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